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TITLE	NEWTON-RAPHSON METHOD FOR DETERMINATION OF POLYNOMIAL ROOTS
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SOURCE LANGUAGE	FOCAL

NEWTON-RAPHSON METHOD FOR DETERMINATION OF POLYNOMIAL ROOTS

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FUNCTION: Determine the 'n' zeroes of a polynomial, f(x), where

$$f(x) = a_0 + a_1x + \ldots + a_nx^n$$

where a_0 and a_n are not equal to zero and a_0, a_1. . ., a_n are in general complex.

METHOD: Newton-Raphson

Given a function in x, f(x), a zero of f(x) may be approximated by repeated iterations through the Newton Raphson approximation, where

$$x_r = x_i - f(x_i)/f'(x_i)$$

x_i : initial estimate for zero of f(x)

$f(x_i)$: value of f(x) at $x = x_i$

$f'(x_i)$: value of first derivative of f(x) at $x = x_i$

x_r : refined estimate for zero of f(x)

Once x_r is evaluated, x_i may be replaced by x_r and another cycle through the iteration initiated, resulting in an even more refined estimate of the zero of f(x). When a sufficiently accurate approximation is obtained, this root may be removed from the polynomial by synthetic division and another root determined. This entire sequence may be repeated until the polynomial has been completely reduced and all the roots determined.

There are two commonly used methods which may be employed to determine when a "sufficiently accurate approximation" for a root has been obtained. One is based on the function value, which approaches zero as 'x' approaches a zero of f(x). Another is based on the fact that for a convergent sequence of Newton-Raphson iterations, the difference in 'x_i' and 'x_r' approaches zero as 'x' approaches a zero of f(x). A method based on the latter principle was chosen. Since 'x_r' and 'x_i' are complex numbers, a relative accuracy may be determined by dividing the absolute value of the difference between the real parts and the imaginary parts of 'x_r' and 'x_i' by the magnitude of 'x_r'. This relative accuracy may be compared to a very small constant and a decision made as to whether or not the sequence should be continued.

USE OF PROGRAM AND POSSIBLE MODIFICATIONS:

The program accepts as input the order of the polynomial, and both the real and imaginary parts of each coefficient. The output consists of the iterations required to reach the desired accuracy for a particular zero, the resulting zero, and the value of the "reduced" function at that point. (The user should note that the "true" function value is printed out only for the initial root. Thereafter the function value is that of the "reduced" function, that is the function obtained by division of the original polynomial by the determined roots. Although a zero of a reduced function may be quite accurate, it may differ significantly from the true zero.) If convergence is not achieved within a specified number of iterations, an appropriate diagnostic is printed and the program halts.

There are cases where Newton-Raphson will not converge with a given starting value and other cases where convergence is unlikely regardless of the starting value. (The most notable of the latter being the case where multiple zeroes are present and the derivative of f(x) as 'x' approaches the zero approaches zero faster than f(x) itself.) In the former case, if convergence is not achieved in the default number of iterations, a different starting value may improve results. This may be affected by altering the values of 'XR' and XI' in line 1.40 of the program (See Example #2). The default value for x_i is 1 + j1. If a complex root is found, the conjugate of that root is the initial guess for the next iteration. (Note that for a polynomial with complex coefficients, roots may or may not occur in conjugate pairs).

The default value for the relative accuracy is 0.00001 and may be modified by changing the value of 'ACC' in line 1.05 of the program. Also in line 1.05 of the program is 'MR', the value of which is the maximum number of iterations allowable before convergence is achieved. (See Example #3)

GLOSSARY OF TERMS --- NEWTON-RAPHSON ROOT FINDER

ACC - VALUE OF RELATIVE ACCURACY

CI( ) - ARRAY CONTAINING IMAGINARY PART OF COEFFICIENTS OF REDUCED POLYNOMIAL

CR( ) - SAME AS ABOVE EXCEPT REAL PART

DI - DERIVATIVE VALUE OF REDUCED POLYNOMIAL (IMAGINARY)

DR - REAL COUNTERPART OF DI

ER - RELATIVE ERROR, TO BE COMPARED TO 'ACC'

GI - FUNCTION VALUE OF REDUCED POLYNOMIAL (IMAGINARY)

GR - REAL COUNTERPART OF GI

IM - IMAGINARY RESULT FROM EITHER COMPLEX MULTIPLICATION OR DIVISION

I1 - IMAGINARY PORTION OF ARGUMENT #1, INPUT TO EITHER COMPLEX MULTIPLICATION OR DIVISION

I2 - SAME AS ABOVE EXCEPT FOR ARGUMENT #2

MR - MAXIMUM NUMBER OF ITERATIONS

N - ORDER OF POLYNOMIAL

RE - REAL COUNTERPART OF 'IM'

RN - ITERATION COUNTER, TO BE COMPARED TO 'MR'

R1 - REAL COUNTERPART OF 'I1'

R2 - REAL COUNTERPART OF 'I2'

TM - TEMPORARY STORAGE LOCATION

XI - INITIAL ESTIMATE OF IMAGINARY PORTION OF ROOT

XR - REAL COUNTERPART OF XI

YI - REFINED ESTIMATE OF 'XI'

YR - REFINED ESTIMATE OF 'XR'

*

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*C                              EXAMPLE # 1
*
*C THIRD ORDER POLYNOMIAL WITH COMPLEX COEFFICIENTS.
*
*C F[X] = X↑3 + (2-J3)*X↑2 + (11-J8)*X + (10-J5)
*




*
*
*GO


ORDER:3




COEFFICIENTS
X↑N  REAL  IMAG
=  3   :1  :0

=  2   :2  :-3

=  1   :11  :-8

=  0   :10  :-5



   ITERATIONS              ROOT                         FUNCTION VALUE
                    REAL PART    IMAG. PART       REAL PART    IMAG. PART
   =     5     =-   1.00000   =    0.00000     =-  0.00001   =    0.00001
   =     8     =-   1.00000   =-   2.00000     =-  0.00000   =    0.00000
   =     0     =    0.00000   =    5.00000     =-  0.00000   =    0.00000*
```

```
*C                              EXAMPLE # 2
*
*C MODIFICATION OF INITIAL ESTIMATE FOR EACH ROOT
*
*M 1.40
S RN=1;A !!"INITIAL ESTIMATE FOR ROOT  REAL",XR," IMAGINARY",XI
*WRITE 1.4
01.40 S RN=1;A !!"INITIAL ESTIMATE FOR ROOT  REAL",XR," IMAGINARY",XI
*
*GO


ORDER:3

COEFFICIENTS
X↑N  REAL  IMAG
=  3   :1  :0
=  2   :2  :-3
=  1   :11  :-8
=  0   :10  :-5

   ITERATIONS           ROOT                         FUNCTION VALUE
                  REAL PART   IMAG. PART       REAL PART   IMAG. PART

INITIAL ESTIMATE FOR ROOT  REAL:-2  IMAGINARY:5

   =    6     =   0.00000  =   5.00000    =   0.00024  =-   0.00012
   =    6     =-  1.00000  =-  2.00000    =-  0.00000  =    0.00000
   =    0     =-  1.00000  =   0.00000    =-  0.00000  =    0.00000*

*GO


ORDER:3

COEFFICIENTS
X↑N  REAL  IMAG
=  3   :1  :0
=  2   :6  :0
=  1   :11  :0
=  0   :6  :0

   ITERATIONS           ROOT                         FUNCTION VALUE
                  REAL PART   IMAG. PART       REAL PART   IMAG. PART

INITIAL ESTIMATE FOR ROOT  REAL:3  IMAGINARY:5

   =   12     =-  1.00000  =-  0.00000    =   0.00000  =    0.00000

INITIAL ESTIMATE FOR ROOT  REAL:-1  IMAGINARY:20

   =   13     =-  2.00000  =   0.00000    =   0.00000  =    0.00000
   =    0     =-  3.00000  =   0.00000    =   0.00000  =    0.00000*
```

```
*C                                  EXAMPLE # 3
*
*C MODIFICATION OF MAXIMUM # OF RUNS & ACCURACY
*
*MODIFY 1.40
S RN=1;S XR=1;S XI=1
*
*
*WRITE 1.05
01.05 S MR=100;S ACC=0.00001
*
*1.05 A !"MAX. RUNS",MR," RELATIVE ACCURACY",ACC
*
*GO

MAX. RUNS:10  RELATIVE ACCURACY:1.0E-6


ORDER:3                 [NOTE: MULTIPLE ROOTS]


COEFFICIENTS
X↑N   REAL   IMAG
=  3   :1  :0
=  2   :3  :0
=  1   :3  :0
=  0   :1  :0


   ITERATIONS              ROOT                           FUNCTION VALUE
                   REAL PART    IMAG. PART        REAL PART    IMAG. PART

CONVERGENCE NOT ACHIEVED AFTER=  10 ITERATIONS*
*
*GO

MAX. RUNS:20  RELATIVE ACCURACY:1.0E-5


ORDER:3

COEFFICIENTS
X↑N   REAL   IMAG
=  3   :1  :0
=  2   :3  :0
=  1   :3  :0
=  0   :1  :0

   ITERATIONS              ROOT                           FUNCTION VALUE
                   REAL PART    IMAG. PART        REAL PART    IMAG. PART
   =  18     =-  0.99376  =    0.00000     =    0.00000  =-   0.00000
   =  13     =-  1.00312  =    0.00542     =    0.00000  =-   0.00000
   =   0     =-  1.00312  =-   0.00542     =    0.00000  =-   0.00000*
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01.05 S MR=100;S ACC=0.00001
01.10 A !!"ORDER",N;I (N-2) 1.1
01.20 T !!"COEFFICIENTS",!,"X↑N   REAL   IMAG"
01.25 F I=0,N;T ! %2,N-I,"  ";A CR(I),CI(I)
01.30 T !!"   ITERATIONS            ROOT
01.31 T "                       FUNCTION VALUE"
01.32 T !"              "
01.33 T "     REAL PART   IMAG. PART
01.34 D 1.33
01.40 S RN=1;S XR=1.27;S XI=0.73
01.50 S GR=CR(0);S GI=CI(0);F I=1,N;D 7
01.60 S DR=N*CR(0);S DI=N*CI(0);F I=1,N-1;D 8
01.70 S R1=GR;S R2=DR;S I1=GI;S I2=DI;D 6
01.75 S YR=XR-RE;S YI=XI-IM;S ER=FSQT(YR*YR+YI*YI)
01.80 S ER=FABS((YR-XR)/ER)+FABS((YI-XI)/ER)
01.85 I (ER-AC) 1.9;S XR=YR;S XI=YI;S RN=RN+1;I (RN-MR) 1.5;G 1.99
01.90 T !,"   "%3,RN,"   ",%08.05,YR,"  ",YI,"    ",GR,"  ",GI
01.92 I (AC-FABS(YI)) 2.1;
01.95 S N=N-1;D 9;I (1-N) 1.4,10.1;Q
01.99 T !!"CONVERGENCE NOT ACHIEVED AFTER",%3,RN," ITERATIONS";D 1.9; Q

02.10 S RN=1;S XR=YR;S XI=-YI;S N=N-1;D 9;I (1-N) 1.5,10.1; Q

05.10 S RE=R1*R2-I1*I2;S IM=I1*R2+I2*R1;R

06.10 S I2=-I2;D 5;S I2=-I2
06.20 S TM=R2*R2+I2*I2;S RE=RE/TM;S IM=IM/TM

07.10 S R1=GR;S R2=XR;S I1=GI;S I2=XI;D 5
07.20 S GR=RE+CR(I);S GI=IM+CI(I);R

08.10 S R1=DR;S R2=XR;S I1=DI;S I2=XI;D 5
08.20 S DR=RE+(N-I)*CR(I);S DI=IM+(N-I)*CI(I);R

09.10 F I=1,N+1;D 9.5;R
09.50 S R1=YR;S R2=CR(I-1);S I1=YI;S I2=CI(I-1);D 5;D 9.6
09.60 S CR(I)=CR(I)+RE;S CI(I)=CI(I)+IM

10.10 S RN=0;S ER=0
10.20 S R1=-CR(1);S R2=CR(0);S I1=-CI(1);S I2=CI(0); D 6
10.30 S YR=RE;S YI=IM;D 1.9; Q
*
```

ADDENDUM TO FOCAL8-64

User experience with this program has indicated that it is necessary to delete the FOCAL extended functions when running FOCAL8-64 on a 4K machine.