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DECUS NO.	8-459
TITLE	TAYEX - TAYLOR EXPANSION EQUATION SOLVER
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SOURCE LANGUAGE	PAL III

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The expression of a function as a truncated Taylor series is a well known calculational technique. In fact, it is the principle method of solution of linear differential equations. The method can easily be extended to the solution of nonlinear differential equations, the solution of transcendental algebraic equations, and integrals.* The program TAYEX can be used to yield approximation solutions to all of these.

The Taylor series of f(x) is written

$$f(x) = f(x)\Big|_{x=x_0} + \sum_{n=1}^{\infty} \frac{1}{n!} \frac{d^{(n)} f(x)}{dx^n}\Bigg|_{x=x_0} (x - x_0)^n$$

Thus, if all of the derivatives of a function are known, the function is known. A truncation after N terms is a good approximation if x is not too far from x_0. Now consider the differential equation in the form

$$Y'' = F(Y', Y, X)$$

Since it is second order there will be two constants in the solution which must be determined by initial conditions. For these we choose y(x) and y(x) and $y(x_0)$. Now find the higher derivatives of y(x) by differentiating y''. Thus, we have the constants:

$$Y(X_0) = C_1$$
$$Y'(X_0) = C_2$$
$$Y''(X_0) = F(Y', Y, X)\Big|_{x=x_0} = F(C_2, C_1, X_0)$$
$$Y'''(X_0) = \frac{d}{dx} F(Y', Y, X)\Big|_{x=x_0}$$
$$Y^{iv}(X_0) = \frac{d^2}{dx^2} F(Y', Y, x)\Big|_{x=x_0}$$

etc.

*The method is mentioned in Rainville, "Elementary Differential Equations", Macmillan Third Edition, page 297.

and so on. The constants c_1 and c_2 are the boundary conditions. In taking the higher derivatives the lower derivatives will appear. Thus, the expression for $y^{(iv)}$ may contain $y^{(iii)}$, y'' and y' as well as y and x_0. However, the iteration from lower derivatives to higher is well defined and straight forward.

Example Solve the harmonic oscillator problem

$$y'' + \frac{k}{m} y = 0$$

for $$y'(0) = 0, \quad y(0) = 1$$

Solution

$$y(0) = 1$$

$$y'(0) = 0$$

$$y''(0) = -\frac{k}{m} y(0) = -\frac{k}{m}$$

$$y'''(0) = -\frac{k}{m} y'(0) = 0$$

$$y^{iv}(0) = -\frac{k}{m} y''(0) = \frac{k^2}{m^2}$$

$$y^{v}(0) = -\frac{k}{m} y'''(0) = 0$$

etc.

Thus, the solution is

$$y = 1 - \frac{\left(\frac{k}{m}\right)}{2!} t^2 + \frac{\left(\frac{k}{m}\right)^2}{4!} t^4 + \cdots$$

$$= \cos \sqrt{k/m}\, t$$

Example: Solve the large displacement pendulum problem with boundary conditions:

$$A\big|_{t=0} = A_0$$

$$\frac{dA}{dt}\bigg|_{t=0} = 0$$

Solution: The equation of motion ~~is~~ for a pendulum of mass m and length r is:

$$m r \frac{d^2A}{dt^2} = -mg \sin A$$

Then

$$A(0) = A_0$$

$$A'(0) = 0$$

$$A''(0) = -\frac{g}{r} \sin A_0$$

$$A'''(0) = -\frac{g}{r} \cos A_0 A_0' = 0$$

$$A^{IV}(0) = -\frac{g}{r}(-\sin A_0 A_0'^2 + \cos A_0 A_0'')$$

$$= \frac{g^2}{r^2} \sin A_0 \cos A_0$$

The answer is

$$A(t) = A_0 - \frac{1}{2!}\frac{g}{r} \sin A_0 t^2 + \frac{1}{4!}\frac{g^2}{r^2} \sin A_0 \cos A_0 t^4 + \cdots$$

which is for small displacements $(\sin A_0 \simeq A_0, \cos A_0 = 1)$

$$A(t) = A_0 \cos \sqrt{\frac{g}{r}}\, t$$

Example: Solve the simultaneous equations:

$$\frac{d^2x}{dt^2} = -k y$$

$$\frac{d^2y}{dt^2} = -m x^2$$

with boundary conditions

$$x(0) = y(0) = 1$$

$$x'(0) = y'(0) = 0$$

Solution:

$$x(0) = 1$$
$$x'(0) = 0$$
$$x''(0) = -k\,y(0) = -k$$
$$x'''(0) = -k\,y'(0) = 0$$
$$x^{IV}(0) = -k\,y''(0) = -k(-m\,x^2(0)) = km$$

etc.

$$y(0) = 1$$
$$y'(0) = 0$$
$$y''(0) = -m\,x^2(0) = -m$$
$$y'''(0) = -m\,2x(0)\,x'(0) = 0$$
$$y^{IV}(0) = -2m\left(x'^2(0) + x(0)\,x''(0)\right)$$
$$= 2mk$$

etc

One pass of TAYEX will have to be for x(t) and one for y(t).

The main restriction to this method is due to the truncation error. In general, for values of the variable close to the point of expansion, the approximation will be good. However, the amount of error will depend on the problem to be solved. Thus it is up to the user to determine the valid range of the expansion. This can sometimes be done by comparing values obtained after n terms and after n-1 terms.

The appended worked example shows the expansion of the function cos x where x is in radians it is calculated for each 0.1 radian between x=0 and x=3.2. Note that the error in the expansion at x=3.14 is about 1%.

```
A:1

B:0

C:-1

D:0

E:1

F:0

G:-1

H:0

I:1

J:0

X:0

DX:.1
LD SR WITH NUM. OF PTS, HIT CONT
+0.1000000E+01
+0.9950040E+00
+0.9800665E+00
+0.9553364E+00
+0.9210609E+00
+0.8775825E+00
+0.8253356E+00
+0.7648422E+00
+0.6967068E+00
+0.6216104E+00
+0.5403029E+00
+0.4535974E+00
+0.3623602E+00
+0.2675033E+00
+0.1699758E+00
+0.7075393E-01
-0.2916860E-01
-0.1287889E+00
-0.2271046E+00
-0.3231239E+00
-0.4158713E+00
-0.5044000E+00
-0.5877938E+00
-0.6651771E+00
-0.7357192E+00
-0.7986345E+00
-0.8531892E+00
-0.8986983E+00
-0.9345233E+00
-0.9600691E+00
-0.9747760E+00
-0.9781084E+00
-0.9695448E+00
```

Operating Instructions

Loading

Load the floating point package and then TAYEX with the binary loader.

Starting

1. Load program with binary loader.
2. Set 0200 in the switch register, press load address.
3. Press START. The program will respond by asking for the derivatives (TAYEX will divide by n!) and initial X value and DX the incrementing constant. The program will then type "LD SR WITH NUM. OF PTS, HIT CONT", load the switch register with the number of points to be calculated (Be sure that this number is converted to OCTAL).
4. Restart at 0200.

```
                /TAYEX
                /
                /PROGRAM TO SOLVE THE TAYLOR EXPANSION EQUATION
                /F(X)=A+(B/1!)X+(C/2!)X↑2+...+(J/9!)X↑9
                /FOR F(X) WHEN X IS GIVEN, X IS INCREMENTED BY DX0
                /BEFORE EACH TIME F(X) IS CALCULATED EXCEPT FOR
                /THE FIRST TIME IT IS CALCULATED.
                /A,B,C,D,E,F,G,H,I,J ARE COEFFICIENTS
                /                             D. PITTS   11/5/70

                FEXT=0000
                FADD=1000
                FMPY=3000
                FDIV=4000
                FGET=5000
                FPUT=6000
                *5
0005  7400              7400
0006  7200              7200
0007  5600              5600
                *200
0200  7300              CLA CLL
0201  6032              KCC
0202  6046              TLS
0203  4775              JMS I CRLFE
0204  1362              TAD KA
0205  4777              JMS I TYPN
0206  4776              JMS I CONIE
0207  4405              JMS I 5        /INPUT A
0210  4407              JMS I 7
0211  6025              FPUT A
0212  0000              FEXT
0213  4775              JMS I CRLFE
0214  1363              TAD KB
0215  4777              JMS I TYPN
0216  4776              JMS I CONIE
0217  4405              JMS I 5        /INPUT B
0220  4407              JMS I 7
0221  6031              FPUT B
0222  0000              FEXT
0223  4775              JMS I CRLFE
0224  1364              TAD KC
0225  4777              JMS I TYPN
0226  4776              JMS I CONIE
0227  4405              JMS I 5        /INPUT C
0230  4407              JMS I 7
0231  4135              FDIV TWO       /DIV. BY 2.
0232  6065              FPUT C
0233  0000              FEXT
0234  4775              JMS I CRLFE
0235  1365              TAD KD
0236  4777              JMS I TYPN
0237  4776              JMS I CONIE
```

```
0240   4405        JMS I 5          /INPUT D
0241   4407        JMS I 7
0242   4141        FDIV SIX         /DIV. BY 6.
0243   6071        FPUT D
0244   0000        FEXT
0245   4775        JMS I CRLFE
0246   1366        TAD KE
0247   4777        JMS I TYPN
0250   4776        JMS I CONIE
0251   4405        JMS I 5          /INPUT E
0252   4407        JMS I 7
0253   4145        FDIV TWFO        /DIV. BY 24.
0254   6075        FPUT E
0255   0000        FEXT
0256   4775        JMS I CRLFE
0257   1367        TAD KF
0260   4777        JMS I TYPN
0261   4776        JMS I CONIE
0262   4405        JMS I 5          /INPUT F
0263   4407        JMS I 7
0264   4151        FDIV ONTW        /DIV. BY 120.
0265   6101        FPUT F
0266   0000        FEXT
0267   4775        JMS I CRLFE
0270   1370        TAD KG
0271   4777        JMS I TYPN
0272   4776        JMS I CONIE
0273   4405        JMS I 5          /INPUT G
0274   4407        JMS I 7
0275   4155        FDIV STW         /DIV. BY 720.
0276   6105        FPUT G
0277   0000        FEXT
0300   4775        JMS I CRLFE
0301   1371        TAD KH
0302   4777        JMS I TYPN
0303   4776        JMS I CONIE
0304   4405        JMS I 5          /INPUT H
0305   4407        JMS I 7
0306   4161        FDIV FIFO        /DIV. BY 5040.
0307   6111        FPUT H
0310   0000        FEXT
0311   4775        JMS I CRLFE
0312   1372        TAD KI
0313   4777        JMS I TYPN
0314   4776        JMS I CONIE
0315   4405        JMS I 5          /INPUT I
0316   4407        JMS I 7
0317   4165        FDIV FOTO        /DIV. BY  40320.
0320   6115        FPUT II
0321   0000        FEXT
0322   4775        JMS I CRLFE
0323   1373        TAD KJ
0324   4777        JMS I TYPN
0325   4776        JMS I CONIE
```

```
0326  4405           JMS I 5          /INPUT J
0327  4407           JMS I 7
0330  4171           FDIV THIS        /DIV. BY 362880.
0331  6121           FPUT J
0332  0000           FEXT
0333  4775           JMS I CRLFE
0334  1374           TAD KX
0335  4777           JMS I TYPN
0336  4776           JMS I CONIE
0337  4405           JMS I 5          /INPUT X (INITIAL VALUE)
0340  4407           JMS I 7
0341  6125           FPUT X
0342  0000           FEXT
0343  4775           JMS I CRLFE
0344  1365           TAD KD
0345  4777           JMS I TYPN
0346  1374           TAD KX
0347  4777           JMS I TYPN
0350  4776           JMS I CONIE
0351  4405           JMS I 5          /INPUT DX (INCREMENTING CONST.)
0352  4407           JMS I 7
0353  6131           FPUT DX0
0354  0000           FEXT
0355  4761           JMS I DIGIM
0356  4760           JMS I EQUA
0357  7402           HLT
0360  0420  EQUA,   EQU
0361  0473  DIGIM,  DIGIN
0362  0301  KA,     301
0363  0302  KB,     302
0364  0303  KC,     303
0365  0304  KD,     304
0366  0305  KE,     305
0367  0306  KF,     306
0370  0307  KG,     307
0371  0310  KH,     310
0372  0311  KI,     311
0373  0312  KJ,     312
0374  0330  KX,     330
0375  0404  CRLFE,  CRLF
0376  0400  CONIE,  CONIN
0377  0412  TYPN,   TYPE
            PAUSE
```

```
                   *400
0400  0000  CONIN,  0
                     TAD K272
0401  1325
0402  4212           JMS TYPE
0403  5600           JMP I CONIN
0404  0000  CRLF,   0
0405  1324           TAD K215
0406  4212           JMS TYPE
0407  1323           TAD K212
0410  4212           JMS TYPE
0411  5604           JMP I CRLF
0412  0000  TYPE,   0
0413  6041           TSF
0414  5213           JMP .-1
0415  6046           TLS
0416  7200           CLA
0417  5612           JMP I TYPE
0420  0000  EQU,    0
0421  4243           JMS CALC       /CALCULATE F(X) AT INITAL X
0422  1322           TAD X0
0423  7041           CIA
0424  3321           DCA X11
0425  1321  NEXT,   TAD X11
0426  1320           TAD M1
0427  3321           DCA X11
0430  4407           JMS I 7
0431  5125           FGET X         /GET X
0432  1131           FADD DX0
0433  6125           FPUT X
0434  0000           FEXT
0435  4243           JMS CALC       /CALCULATE F(X) AT X+DX0
0436  7300           CLA CLL
0437  1321           TAD X11
0440  7640           SZA CLA        /CALCULATE F(X) AGAIN?
0441  5225           JMP NEXT       /YES
0442  5620           JMP I EQU      /NO
0443  0000  CALC,   0
0444  4407           JMS I 7
0445  5121           FGET J         /GET J
0446  3125           FMPY X         /MULTIPLY X
0447  1115           FADD II        /ADD I
0450  3125           FMPY X         /MULTIPLY X
0451  1111           FADD H         /ADD H
0452  3125           FMPY X         /MULTIPLY X
0453  1105           FADD G         /ADD G
0454  3125           FMPY X         /MULTIPLY X
0455  1101           FADD F         /ADD F
0456  3125           FMPY X         /MULTPLY X
0457  1075           FADD E         /ADD E
0460  3125           FMPY X         /MULTIPLY X
0461  1071           FADD D         /ADD D
0462  3125           FMPY X         /MULTIPLY X
```

```
0463  1065             FADD C          /ADD C
0464  3125             FMPY X          /MULTIPLY X
0465  1031             FADD B          /ADD B
0466  3125             FMPY X          /MULTIPLY X
0467  1025             FADD A          /ADD A
0470  0000             FEXT
0471  4406             JMS I 6         /OUTPUT F(X)
0472  5643             JMP I CALC
0473  0000  DIGIN,   0
0474  7300             CLA CLL
0475  4302             JMS MESS
0476  7402             HLT
0477  7604             LAS
0500  3322             DCA X0
0501  5673             JMP I DIGIN
0502  0000  MESS,    0
0503  7300             CLA CLL
0504  6046             TLS
0505  1326             TAD CHARAC
0506  3010             DCA IR1
0507  1316             TAD M40
0510  3317             DCA COUNT
0511  1410  NXT,     TAD I IR1
0512  4212             JMS TYPE
0513  2317             ISZ COUNT
0514  5311             JMP NXT
0515  5702             JMP I MESS

0516  7736  M40,     -42
0517  0000  COUNT,   0
0520  0001  M1,      1
0521  0000  X11,     0
0522  0000  X0,      0
0523  0212  K212,    212
0524  0215  K215,    215
0525  0272  K272,    272
0526  0526  CHARAC,  .
0527  0314           314
0530  0304           304
0531  0240           240
0532  0323           323
0533  0322           322
0534  0240           240
0535  0327           327
0536  0311           311
0537  0324           324
0540  0310           310
0541  0240           240
0542  0316           316
0543  0325           325
0544  0315           315
0545  0256           256
0546  0240           240
0547  0317           317
```

108	0550	0306		306
109	0551	0240		240
110	0552	0320		320
111	0553	0324		324
112	0554	0323		323
113	0555	0254		254
114	0556	0240		240
115	0557	0310		310
116	0560	0311		311
117	0561	0324		324
118	0562	0240		240
119	0563	0303		303
120	0564	0317		317
121	0565	0316		316
122	0566	0324		324
123	0567	0212		212
124	0570	0215		215
125			IR1=10	
126			PAUSE	

```
                  / STORAGE LOCATIONS FOR FLOATING PT. VARIABLES
                  *25
0025    0000      A,      0
0026    0000              0
0027    0000              0
0030    0000              0
0031    0000      B,      0
0032    0000              0
0033    0000              0
0034    0000              0
                  *65
0065    0000      C,      0
0066    0000              0
0067    0000              0
0070    0000              0
0071    0000      D,      0
0072    0000              0
0073    0000              0
0074    0000              0
0075    0000      E,      0
0076    0000              0
0077    0000              0
0100    0000              0
0101    0000      F,      0
0102    0000              0
0103    0000              0
0104    0000              0
0105    0000      G,      0
0106    0000              0
0107    0000              0
0110    0000              0
0111    0000      H,      0
0112    0000              0
0113    0000              0
0114    0000              0
0115    0000      II,     0
0116    0000              0
0117    0000              0
0120    0000              0
0121    0000      J,      0
0122    0000              0
0123    0000              0
0124    0000              0
0125    0000      X,      0
0126    0000              0
0127    0000              0
0130    0000              0
0131    0000      DX0,    0
0132    0000              0
0133    0000              0
0134    0000              0
                  /  STORAGE LOCATIONS FOR CONSTANTS
0135    0002      TWO,    0002
```

```
0136  2000            2000
0137  0000            0000
0140  0000            0000
0141  0003  SIX,      0003
0142  3000            3000
0143  0000            0000
0144  0000            0000
0145  0005  TWFO,     0005
0146  3000            3000
0147  0000            0000
0150  0000            0000
0151  0007  ONTW,     0007
0152  3600            3600
0153  0000            0000
0154  0000            0000
0155  0012  STW,      0012
0156  2640            2640
0157  0000            0000
0160  0000            0000
0161  0015  FIFO,     0015
0162  2354            2354
0163  0000            0000
0164  0000            0000
0165  0020  FOTO,     0020
0166  2354            2354
0167  0000            0000
0170  0000            0000
0171  0023  THIS,     0023
0172  2611            2611
0173  4000            4000
0174  0000            0000

A        0025
B        0031
C        0065
CALC     0443
CHARAC   0526
CONIE    0376
CONIN    0400
COUNT    0517
CRLF     0404
CRLFE    0375
D        0071
DIGIM    0361
DIGIN    0473
DX0      0131
E        0075
EQU      0420
EQUA     0360
F        0101
FIFO     0161
FOTO     0165
G        0105
H        0111
II       0115
```

IR1	0010
J	0121
KA	0362
KB	0363
KC	0364
KD	0365
KE	0366
KF	0367
KG	0370
KH	0371
KI	0372
KJ	0373
KX	0374
K212	0523
K215	0524
K272	0525
MESS	0502
M1	0520
M40	0516
NEXT	0425
NXT	0511
ONTW	0151
SIX	0141
STW	0155
THIS	0171
TWFO	0145
TWO	0135
TYPE	0412
TYPN	0377
X	0125
X0	0522
X11	0521