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DECUS NO.	FOCAL8-68
TITLE	DETERMINATION OF ROOTS OF A POLYNOMIAL
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SOURCE LANGUAGE	FOCAL

Program Writeup-Determination of Roots of a Polynomial

The overall program has three main parts. These are:

1. Search for a real root.
2. Evaluation of real root if it is found.
3. Solution of complex roots if no real roots are found.

In the 8k version all these procedures are carried out within the program with no need for user intervention. The 4k version has four individual programs. Program I carries out parts 1 and 2 above. Program II is a polynomial division subroutine which is used to reduce the order of the polynomial, following which program I can be used on the reduced polynomial. Program III solves for the complex roots. Program IV is another polynomial division sub-routine which is used, if real roots are turned up in the solution for complex roots, to reduce the order of the polynomial by two, and so return to the beginning of program I again.

Since the 4k and 8k versions are identical in all significant aspects, the following discussion is pertinent to both versions. We attempt here to guide the user so he can adapt the program to his specific needs.

The search for a real root uses a brute force method which is very fast for finding well-separated real roots, but very slow in determining the probable existence of complex roots. We use the fact that all the roots of a polynomial are less than the sum of the absolute values of the polynomial coefficients. This value is computed as variable ST, and is used to set-up the search. First the negative real axis, then the positive axis, is searched in the intervals ST/2. If no function crossing is detected, the search interval is divided by 10, and if no function crossing still detected, the interval is divided by 10 again. This last search is time-consuming, and can be omitted simply by changing the argument of the IF statement in line 2.55 to be (T-2) rather than (T-3). A finer, but more time-consuming, search can be had by using (T-4).

The subroutine of group 7 calculates the value of the polynomial as required during the search and real root evaluation.

If the search uncovers an axis crossing, the program enters an interval halving technique, (group 4 statements) to obtain an estimate of the root to a tolerance of 0.1. Then the program uses a Newton-Raphson technique (group 5 statements) to determine the root to a tolerance of 0.00001.

At this point program I of the 4k version terminates. The user must use program II to divide the root into the original polynomial to obtain a polynomial of degree one less. The reduced polynomial can then be operated on by program I again, the process continuing until all real roots have been extracted, down to a polynomial of degree two.

Complex roots are solved for by Bairstow's method. Here the program given in reference 1 is closely followed. The programmed tolerance is 0.0001, and is set in line 2.82 of the 4k version, and in line 16.82 of the 8k version. (Groups 2 and 3 of the 4k version are equivalent to groups 16 and 17 of the 8k version.) The polynomial division sub-routine (groups 12, 13 and 14) of the complex roots portion is fundamental to the Bairstow method as programmed, and is also used in the 8k version to reduce the order of the polynomial as roots, either real or complex, are extracted.

As given, the 4k version will solve for all the complex roots of polynomials of order 4 and 6. By paring away some type-out statements it may be possible to extend this to handle order 8. (Storage in the FOCAL system is the limiting factor in all these programs.)

Polynomials of order 2 must be solved by applying the quadratic formula. This has not been supplied in the 4k version (see sample program in FOCAL manual), but is incorporated in the 8k version.

It may happen that the initial search will not uncover real roots that are very close together. Bairstow's method will then result in a second-order polynomial whose roots are a pair of real roots of the original polynomial. The 8k version has been programmed to recognize this case, and following extraction of the pair of real roots, will divide them out and proceed to search anew the reduced polynomial. The 4k user will recognize this case when he gets error message ?30.48 at line 3.10. He can get the coefficients of the quadratic by examining the values of the variables P and U at that point. The resulting polynomial, x^2+Px+U, can be solved by the quadratic formula. Program IV can be used with the second-order polynomial above to divide it out of the given polynomial, and the new polynomial, reduced by order two, can be solved from the beginning again.

REFERENCES:

1. Introductory Computer Methods and Numerical analysis, Ralph H. Pennington, Macmillan, 1965

2. Numerical Methods for Scientists and Engineers, R.W. Hamming, McGraw-Hill, 1962

3. Introduction to Numerical Analysis, F.B. Hildebrand, McGraw-Hill, 1956

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Ø1.1Ø  T   !! "REAL ROOTS OF POLYNOMIAL - PART 1
Ø1.3Ø  A   ! "ORDER OF POLYNOMIAL ", N
Ø1.4Ø  T   !! "TYPE POLYNOMIAL COEFFS", !
Ø1.45  F   I=1, N+1;A AM(I)
Ø1.5Ø  D   1.1
Ø1.55  T   !! "POLYNOMIAL COEFFS", !
Ø1.6Ø  F   I=1, N+1;T %, AM(I), !
Ø1.7Ø  S   ST=Ø
Ø1.8Ø  F   I=1, N+1;S ST=ST+FABS(AM(I))

Ø2.Ø5  S   XS=-ST/2;S T=Ø
Ø2.1Ø  S   X2=XS;S X1=Ø
Ø2.15  D   3
Ø2.2Ø  I   (Y1*Y2)4.1,3.25,2.25
Ø2.25  S   X2=X2+XS;S X1=X1+XS
Ø2.3Ø  I   (FABS(X2)-ST)2.15,2.15,2.4
Ø2.4Ø  I   (X2)2.5,2.5,2.55
Ø2.5Ø  S   XS=-XS;G 2.1
Ø2.55  S   T=T+1;I (T-3)2.57,2.6,2.6
Ø2.57  S   XS=-XS/1Ø;G 2.1
Ø2.6Ø  T   !! "FUNCTION VALUE HAS NOT CROSSED REAL AXIS DURING
Ø2.6Ø  T   !!  "PRELIMINARY SEARCH - ROOTS MAY BE COMPLEX", !;Ø

Ø3.1Ø  S   X=X1;D 7
Ø3.15  S   Y1=Z;S X=X2;D 7
Ø3.2Ø  S   Y2=Z;R
Ø3.25  T   !! "ONE ROOT IS  "
Ø3.3Ø  IF  (Y2)3.5,3.4,3.5
Ø3.4Ø  T   X2;S XR=X2;G 6.1
Ø3.5Ø  T   X1;S XR=X1;G 6.1

Ø4.1Ø  S   X=X2;D 7
Ø4.12  S   Y=Z
Ø4.13  S   X=X1
Ø4.15  D   7
Ø4.2Ø  IF  (FABS(X2-X)-.1)5.1,5.1,4.25
Ø4.25  IF  (Y*Z)4.3,4.6,4.4
Ø4.3Ø  S   X1=X;G 4.5
Ø4.4Ø  S   X2=X
Ø4.5Ø  S   X=(X1+X2)/2;G 4.15
Ø4.6Ø  D   3.25;T  X;S XR=X;G 6.1
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Ø5.1Ø   S   XC=X
Ø5.15   F   I=1,N;S AP(I)=AM(I)*(N-I+1)
Ø5.2Ø   S   X=XC;D 7
Ø5.25   S   YA=Z
Ø5.3Ø   S   YB=AP(1);F I=1,N-1;S YB=YB*X+AP(I+1)
Ø5.4Ø   S   XN=XC-YA/YB
Ø5.5Ø   I   (FABS(XN-XC)-.ØØØØ1)5.7,5.7,5.6
Ø5.6Ø   S   XC=XN;G 5.2
Ø5.7Ø   D   3.25;T XN;S XR=XN;Q

Ø7.1Ø   S   Z=AM(1)
Ø7.2Ø   F   I=1,N;S Z=Z*X+AM(I+1)
Ø7.3Ø   R
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11.Ø5   T    !! "ROOTS OF POLYNOMIAL - PART II
11.1Ø   T    !! "POLYNOMIAL DIVISION FOR REAL ROOT REDUCTION
11.2Ø   A    ! "ORDER OF POLYNOMIAL   ",N
11.3Ø   S    M=1;S B(1)=1
11.4Ø   T    !! "TYPE IN POLYNOMIAL COEFFS",!
11.45   F    I=1,N+1;A A(I)
11.5Ø   A    ! "TYPE IN VALUE OF REAL ROOT   ",RR
11.55   S    B(2)=-RR
11.6Ø   T    !! "POLYNOMIAL COEFFS",!
11.65   F    I=1,N+1;T  %,A(I), !
11.7Ø   T    !! "VALUE OF REAL ROOT   ",RR,!

12.1Ø   S    LL=N+1
12.2Ø   F    I=1,LL;S G(I)=A(I)
12.3Ø   S    LL=N-M+1
12.4Ø   F    I=1,LL;D 13
12.5Ø   F    I=1,M;S R(I)=G(N-M+I+1)
12.6Ø   S    L=M-1
12.7Ø   IF   (R(1)) 12.9,12.75,12.9
12.75   IF   (L) 12.9,12.9,12.77
12.77   F    I=1,L;S R(I)=R(I+1)
12.8Ø   S    L=L-1;G 12.7
12.9Ø   G    15.1

13.1Ø   S    Q(I)=G(I)/B(1)
13.2Ø   F    J=1,M;D 14
13.3Ø   R

14.1Ø   S    W=Q(I)*B(J+1)
14.2Ø   S    G(I+J)=G(I+J)-W
14.3Ø   IF   (FABS(G(I+J))-.ØØØ1*FABS(W)) 14.4,14.4,14.5
14.4Ø   S    G(I+J)=Ø
14.5Ø   R

15.1Ø   T    !! "RESULTS OF DIVISION
15.2Ø   T    !! "REDUCED POLYNOMIAL",!
15.25   F    I=1,LL;T  %,Q(I), !
15.4Ø   Q
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Ø1.Ø5   T   !! "ROOTS OF POLYNOMIAL - PART III - COMPLEX ROOTS
Ø1.3Ø   A   ! "ORDER OF POLYNOMIAL  ",N
Ø1.4Ø   T   !! "TYPE POLYNOMIAL COEFFS",!
Ø1.45   F   I=1,N+1;A AM(I)
Ø1.55   T   !! "POLYNOMIAL COEFFS",!
Ø1.6Ø   F   I=1,N+1;T  %,AM(I),!

Ø2.1Ø   S   B(1)=1;S B(2)=Ø;S B(3)=Ø;S M=2
Ø2.2Ø   F   I=1,N+1;S  G(I)=AM(I)
Ø2.3Ø   D   12
Ø2.35   F   I=1,N-1;S QM(I)=Q(I)
Ø2.37   F   I=1,2;S RM(I)=R(I)
Ø2.4Ø   I   (RM(1)) 2.45,2.95,2.45
Ø2.45   S   QM(N)=RM(1);S QM(N+1)=RM(2)
Ø2.5Ø   F   I=1,N+1;S G(I)=QM(I)
Ø2.55   D   12
Ø2.57   F   I=1,N-1;S C(I)=Q(I)
Ø2.58   F   I=1,2;S T(I)=R(I)
Ø2.6Ø   S   D=C(N-1)*(C(N-1)-B(2)*C(N-2))
Ø2.65   S   D=D-C(N-2)*(T(1)-RM(1)-B(2)*C(N-1))
Ø2.7Ø   IF  (D) 2.75,2.72,2.75
Ø2.72   S   B(2)=B(2)+.1;S B(3)=B(3)+.1;G 2.85
02.75   S   D2=(RM(1)*(C(N-1)+B(2)*C(N-2))-RM(2)*C(N-2))/D
Ø2.77   S   D3=(C(N-1)*RM(2)-RM(1)*(T(1)-RM(1)+B(2)*C(N-1)))/D
Ø2.8Ø   S   B(2)=B(2)+D2;S B(3)=B(3)+D3
Ø2.81   S   X=FABS(D2)+FABS(D3);S Y=FABS(B(2))+FABS(B(3))
Ø2.82   I   (X/Y-.ØØØ1)2.95,2.95,2.2
Ø2.95   S   P=B(2);S U=B(3)

Ø3.1Ø   S   RP=-P/2;S IM=FSQT(U-P↑2/4)
Ø3.25   T   !! "ROOTS ARE - REAL PART  ",RP
Ø3.26   T   !  "IMAGINARY PART  ",IM
Ø3.28   I   (2-N)3.3;Q
Ø3.3Ø   F   I=1,N+1; S G(I)=AM(I)
Ø3.4Ø   D   12
Ø3.5Ø   S   N=N-2;I (N-2)3.8,3.8,3.55
03.55   F   I=1,N+1;S AM(I)=Q(I)
Ø3.57   G   2.1
Ø3.8Ø   S   P=Q(2);S U=Q(3);G 3.1
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12.30   S    LL=N-M+1
12.40   F    I=1,LL;D 13
12.50   F    I=1,M;S R(I)=G(N-M+I+1)
12.60   S    L=M-1
12.70   IF   (R(1)) 12.9,12.75,12.9
12.75   IF   (L) 12.9,12.9,12.77
12.77   F    I=1,L;S R(I)=R(I+1)
12.80   S    L=L-1;G 12.7
12.90   R

13.10   S    Q(I)=G(I)/B(1)
13.20   F    J=1,M;D 14
13.30   R

14.10   S    W=Q(I)*B(J+1)
14.20   S    G(I+J)=G(I+J)-W
14.30   IF   (FABS(G(I+J))-.0001*FABS(W)) 14.4,14.4,14.5
14.40   S    G(I+J)=0
14.50   R
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11.Ø5   T   !! " ROOTS OF POLYNOMIAL - PART IV
11.1Ø   T   !! "POLYNOMIAL DIVISION FOR REAL ROOT REDUCTION
11.2Ø   A   ! "ORDER OF POLYNOMIAL  ",N
11.3Ø   S   M=2;S B(1)=1
11.4Ø   T   !! "TYPE IN POLYNOMIAL COEFFS",!
11.45   F   I=1,N+1;A A(I)
11.5Ø   A   ! "TYPE IN COEFFS, P, U, OF X↑2+PX+U   ",P,U
11.55   S   B(2)=P;S B(3)=U
11.6Ø   T   !! "POLYNOMIAL COEFFS",!
11.65   F   I=1,N+1;T  %,A(I), !
11.7Ø   T   !! "P   ",P,"   U   ",U

12.1Ø   S   LL=N+1
12.2Ø   F   I=1,LL;S G(I)=A(I)
12.3Ø   S   LL=N-M+1
12.4Ø   F   I=1,LL;D 13
12.5Ø   F   I=1,M;S R(I)=G(N-M+I+1)
12.6Ø   S   L=M-1
12.7Ø   IF  (R(1)) 12.9,12.75,12.9
12.75   IF  (L) 12.9,12.9,12.77
12.77   F   I=1,L;S R(I)=R(I+1)
12.8Ø   S   L=L-1;G 12.7
12.9Ø   G   15.1

13.1Ø   S   Q(I)=G(I)/B(1)
13.2Ø   F   J=1,M;D 14
13.3Ø   R

14.1Ø   S   W=Q(I)*B(J+1)
14.2Ø   S   G(I+J)=G(I+J)-W
14.3Ø   IF  (FABS(G(I+J))-.ØØØ1*FABS(W)) 14.4,14.4,14.5
14.4Ø   S   G(I+J)=Ø
14.5Ø   R

15.1Ø   T   !! "RESULTS OF DIVISION
15.2Ø   T   !! "REDUCED POLYNOMIAL",!
15.25   F   I=1,LL;T  %,Q(I), !
15.4Ø   Q
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Ø1.1Ø   T   !! " ROOTS OF POLYNOMIAL
Ø1.3Ø   A   ! "ORDER OF POLYNOMIAL  ",N
Ø1.4Ø   T   !! "TYPE POLYNOMIAL COEFFS",!
Ø1.45   F   I=1,N+1;A AM(I)
Ø1.5Ø   D   1.1
Ø1.55   T   !! "POLYNOMIAL COEFFS",!
Ø1.6Ø   F   I=1,N+1;T  %,AM(I),!
Ø1.7Ø   S   ST=Ø
Ø1.8Ø   F   I=1,N+1;S ST=ST+FABS(AM(I))

Ø2.Ø5   S   XS=-ST/2;S T=Ø
Ø2.1Ø   S   X2=XS;S X1=Ø
Ø2.15   D   3
Ø2.2Ø   I   (Y1*Y2)4.1,3.25,2.25
Ø2.25   S   X2=X2+XS;S X1=X1+XS
Ø2.3Ø   I   (FABS(X2)-ST)2.15,2.15,2.4
Ø2.4Ø   I   (X2)2.5,2.5,2.55
Ø2.5Ø   S   XS=-XS;G 2.1
Ø2.55   S   T=T+1;I (T-3)2.57,2.6,2.6
Ø2.57   S   XS=-XS/1Ø;G 2.1
Ø2.6Ø   T   !! "FUNCTION VALUE HAS NOT CROSSED REAL AXIS DURING
Ø2.65   T   ! "SEARCH - ROOTS MAY BE COMPLEX",!;G 16.Ø1

Ø3.1Ø   S   X=X1;D 7
Ø3.15   S   Y1=Z;S X=X2;D 7
Ø3.2Ø   S   Y2=Z;R
Ø3.25   T   !! "ONE REAL ROOT IS   "
Ø3.3Ø   IF  (Y2)3.5,3.4,3.5
Ø3.4Ø   T   X2;S XR=X2;G 6.1
Ø3.5Ø   T   X1;S XR=X1;G 6.1

Ø4.1Ø   S   X=X2;D 7
Ø4.12   S   Y=Z
04.13   S   X=X1
Ø4.15   D   7
Ø4.2Ø   IF  (FABS(X2-X)-.1)5.1,5.1,4.25
Ø4.25   IF  (Y*Z)4.3,4.6,4.4
Ø4.3Ø   S   X1=X;G 4.5
Ø4.4Ø   S   X2=X
Ø4.5Ø   S   X=(X1+X2)/2;G 4.15
Ø4.6Ø   D   3.25;T X,!;S XR=X;G 6.1
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Ø5.1Ø   S    XC=X
Ø5.15   F    I=1, N;S AP(I)=AM(I)*(N-I+1)
Ø5.2Ø   S    X=XC;D7
Ø5.25   S    YA=Z
Ø5.3Ø   S    YB=AP(1);F I=1, N-1;S YB=YB*X+AP(I+1)
Ø5.4Ø   S    XN=XC-YA/YB
Ø5.5Ø   I    (FABS(XN-XC)-.ØØØØ1)5.7,5.7,5.6
Ø5.6Ø   S    XC=XN;G 5.2
Ø5.7Ø   D    3.25;T XN,!;S XR=XN

Ø6.1Ø   S    M=1;S B(1)=1;S B(2)=-XR
Ø6.2Ø   F    I=1, N+1;S G(I)=AM(I)
Ø6.3Ø   D    12
Ø6.4Ø   F    I=1, N;S AM(I)=Q(I)
Ø6.55   S    N=N-1;I (N-2)6.6,6.6,1.7
Ø6.6Ø   S    RQ=AM(2)*AM(2)-4*AM(1)*AM(3)
Ø6.65   IF   (RQ)6.8,6.7,6.7
Ø6.7Ø   S    Q1=(-AM(2)+FSQT(RQ))/(2*AM(1))
Ø6.71   S    Q2=(-AM(2)-FSQT(RQ))/(2*AM(1))
Ø6.75   T    !! "ROOTS OF QUADRATIC ARE  "
Ø6.77   T    Q1, "    ",Q2,!;Q
Ø6.8Ø   S    QR=-AM(2)/(2*AM(1))
Ø6.81   S    QI=FSQT(AM(3)-AM(2)*AM(2)/4*AM(1))
Ø6.82   D    6.75
Ø6.85   T    ! "REAL PART   ",QR, "    IMAG PART    ",QI,!;Q

Ø7.1Ø   S    Z=AM(1)
Ø7.2Ø   F    I=1, N;S Z=Z*X+AM(I+1)
Ø7.3Ø   R

12.3Ø   S    LL=N-M+1
12.4Ø   F    I=1, LL;D 13
12.5Ø   F    I=1, M;S R(I)=G(N-M+I+1)
12.6Ø   S    L=M-1
12.7Ø   IF   (R(1)) 12.9,12.75,12.9
12.75   IF   (L) 12.9,12.9,12.77
12.77   F    I=1, L;S R(I)=R(I+1)
12.8Ø   S    L=L-1;G 12.7
12.9Ø   R

13.1Ø   S    Q(I)=G(I)/B(1)
13.2Ø   F    J=1, M;D 14
13.3Ø   R

14.1Ø   S    W=Q(I)*B(J+1)
14.2Ø   S    G(I+J)=G(I+J)-W
14.3Ø   IF   (FABS(G(I+J))-.ØØØ1*FABS(W)) 14.4,14.4,14.5
14.4Ø   S    G(I+J)=Ø
14.5Ø   R
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16.01   T    !! "BEGIN SOLUTION FOR COMPLEX ROOTS
16.05   S    IR=1
16.10   S    B(1)=1;S B(2)=0;S B(3)=0;S M=2;S  JI=0
16.20   F    I=1,N+1;S G(I)=AM(I)
16.30   D    12
16.35   F    I=1,N-1;S QM(I)=Q(I)
16.37   F    I=1,2;S RM(I)=R(I)
16.40   IF   (RM(1)) 16.45,16.95,16.45
16.45   S    QM(N)=RM(1);S QM(N+1)=RM(2)
16.50   F    I=1,N+1;S G(I)=QM(I)
16.55   D    12
16.57   F    I=1,N-1;S C(I)=Q(I)
16.58   F    I=1,2;S T(I)=R(I)
16.60   S    D=C(N-1)*(C(N-1)-B(2)*C(N-2))
16.65   S    D=D-C(N-2)*(T(1)-RM(1)-B(2)*C(N-1))
16.70   IF   (D) 16.75,16.72,16.75
16.72   S    B(2)=B(2)+.1;S B(3)=B(3)+.1;G 16.85
16.75   S    D2=(RM(1)*(C(N-1)+B(2)*C(N-2))-RM(2)*C(N-2))/D
16.77   S    D3=(C(N-1)*RM(2)-RM(1)*(T(1)-RM(1)+B(2)*C(N-1)))/D
16.80   S    B(2)=B(2)+D2;S B(3)=B(3)+D3
16.81   S    X=FABS(D2)+FABS(D3);S Y=FABS(B(2))+FABS(B(3))
16.82   IF   (X/Y-.0001)  16.95, 16.95, 16.85
16.85   S    JI=JI+1
16.87   IF   (100-J)16.9,16.2,16.2
16.90   T    !!! "NO CONVERGENCE AFTER 100 ITERATIONS",Q
16.95   S    P=B(2);S U=B(3)

17.05   S    T1=(P↑2-4*U);I (T1)17.1,17.6,17.6
17.10   S    RP=-P/2;S IM=FSQT(U-P↑2/4)
17.20   T    !!! "COMPLEX ROOTS SET   ",%3.0,IR,%
17.25   T    !! "REAL PART      ",RP
17.26   T    ! "IMAGINARY PART     ",IM,!
17.28   IF   (N-2) 17.9,17.9,17.3
17.30   F    I=1,N+1; S G(I)=AM(I)
17.40   D    12
17.45   S    IR=IR+1
17.50   S    N=N-2; IF (N-2)17.8,17.8,17.55
17.55   F    I=1,N+1;S AM(I)=Q(I)
17.57   G    16.1
17.60   S    T1=FSQT(T1);S T2=(-P+T1)/2;S T3=(-P-T1)/2
17.65   D    3.25;T T2,!;D 3.25;T T3,!
17.70   F    I=1,N+1;S G(I)=AM(I)
17.72   D    12
17.74   F    I=1,N-1; S AM(I)=Q(I)
17.76   S    N=N-2;G 1.7
17.80   S    P=Q(2);S U=Q(3);G 17.1
17.90   Q
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